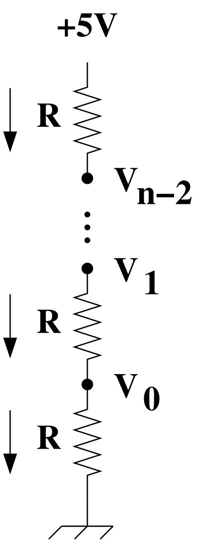
Embedded Systems (CS [45]163)
Homework 1 Solutions

February 14, 2010

Question 1

Consider the following circuit that is composed of n resistors:



1. (20pts) Derive equations for V_i. Hint: you may use your knowledge of series resistors in your derivation.

   For any point V_i, we can combine the resistors below into a single resistor of resistance $(i+1) \times R\ \Omega$. Likewise, the resistors above can be combined into a single resistor of $(n-i-1) \times R\ \Omega$. Furthermore, the resistance of the entire series is $n \times R\ \Omega$.

   By Ohm's law, combined with Kirchoff's current law, we know that the current through the circuit is:

$$I = (5-0)/nR$$

   and

$$V_i - 0 = (i+1)RI$$

   Therefore:

$$\begin{aligned} V_i &= (i+1)R5/nR \\ &= 5\frac{i+1}{n} \end{aligned}$$

2. (5pts) Assume that $n = 5$, what are the four V's?

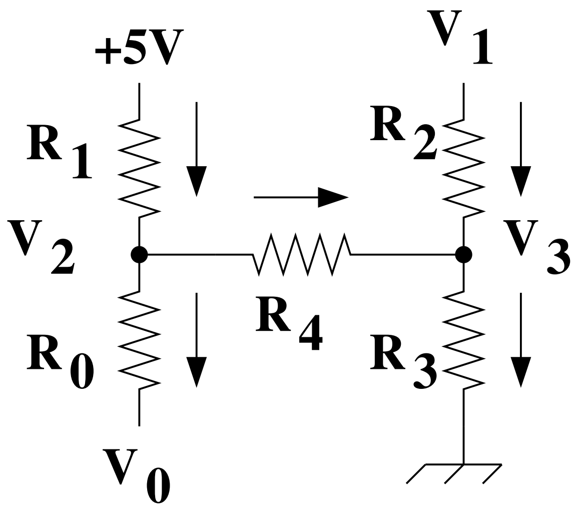
$$\begin{aligned} V_0 &= 1V \\ V_1 &= 2V \\ V_2 &= 3V \\ V_3 &= 4V \end{aligned}$$

3. (5pts) Assume that $n = 10$, what are the nine V's?

$$\begin{aligned}
V_0 &= 0.5V \\
V_1 &= 1V \\
V_2 &= 1.5V \\
V_3 &= 2V \\
V_4 &= 2.5V \\
V_5 &= 3V \\
V_6 &= 3.5V \\
V_7 &= 4V \\
V_8 &= 4.5V
\end{aligned}$$

Question 2

Consider the following circuit:



Assume that V_0, V_1 and R_i are given.

1. (10pts) List the fundamental equations that are derived directly from the circuit (note that you should have the same number of equations as unknowns).

   These are always true:

$$\begin{aligned}
5 - V_2 &= I_1 R_1 \\
V_2 - V_0 &= I_0 R_0 \\
V_2 - V_3 &= I_4 R_4 \\
V_1 - V_3 &= I_2 R_2 \\
V_3 - 0 &= I_3 R_3 \\
I_1 &= I_0 + I_4 \\
I_2 + I_4 &= I_3
\end{aligned}$$

2. (20pts) Derive equations for V_2 and V_3. One of the two should be a function of only known variables; the other may assume knowledge of the first.

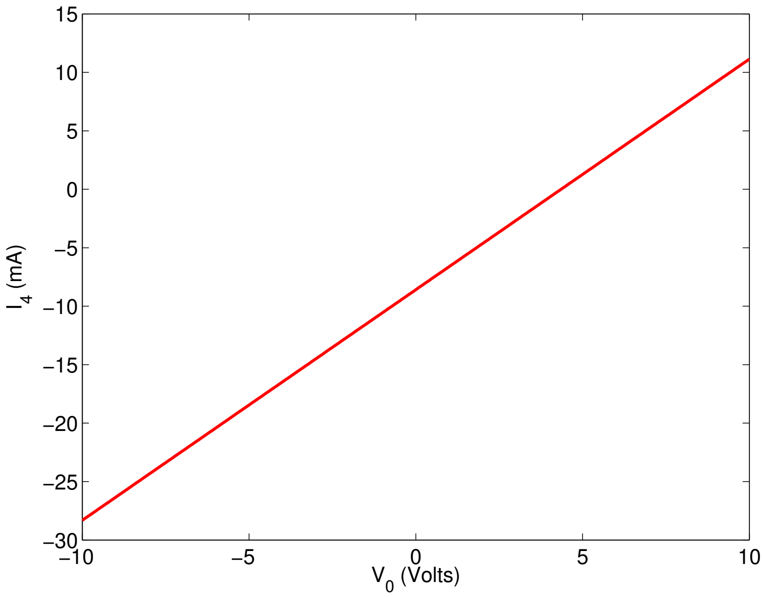
$$
\begin{aligned}
\frac{5-V_2}{R_1} &= \frac{V_2-V_0}{R_0}+\frac{V_2-V_3}{R_4} \\
R_0R_4(5-V_2) &= R_1R_4(V_2-V_0) \\
&\quad +R_0R_1(V_2-V_3) \\
5R_0R_4-R_0R_4V_2 &= R_1R_4V_2-R_1R_4V_0 \\
&\quad +R_0R_1V_2-R_0R_1V_3 \\
5R_0R_4+R_1R_4V_0+R_0R_1V_3 &= (R_0R_4+R_1R_4+R_0R_1)V_2 \\
5R_0R_4+R_1R_4V_0+R_0R_1V_3 &= R_AV_2 \\
\frac{5R_0R_4+R_1R_4V_0+R_0R_1V_3}{R_A} &= V_2 \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{V_1-V_3}{R_2}+\frac{V_2-V_3}{R_4} &= \frac{V_3}{R_3} \\
R_3R_4(V_1-V_3)+R_2R_3(V_2-V_3) &= R_2R_4V_3 \\
R_3R_4V_1-R_3R_4V_3+R_2R_3V_2-R_2R_3V_3 &= R_2R_4V_3 \\
R_3R_4V_1+R_2R_3V_2 &= (R_3R_4+R_2R_3+R_2R_4)V_3 \\
R_3R_4V_1+R_2R_3V_2 &= R_BV_3
\end{aligned}
$$

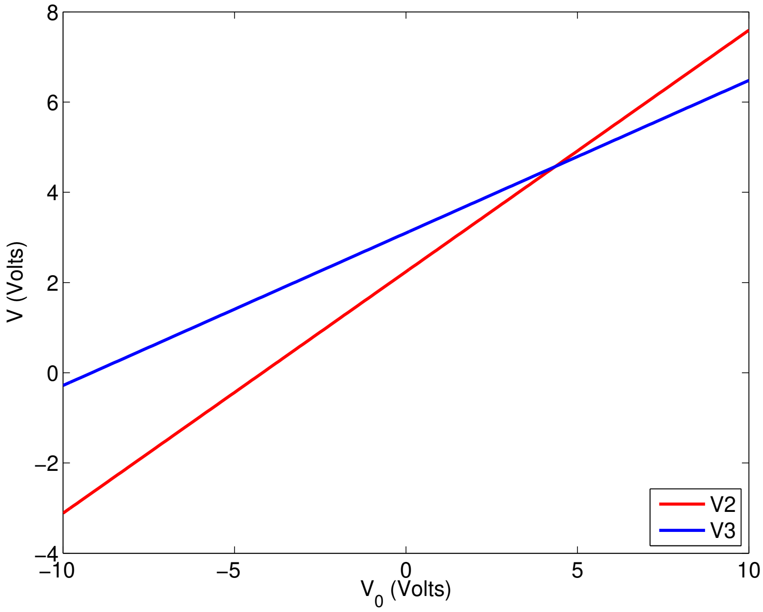
$$
\begin{aligned}
R_3R_4R_AV_1+R_2R_3(5R_0R_4+R_1R_4V_0+R_0R_1V_3) &= R_AR_BV_3 \\
R_3R_4R_AV_1+R_2R_3(5R_0R_4+R_1R_4V_0) &= (R_AR_B-R_0R_1R_2R_3)V_3 \\
\frac{R_3R_4R_AV_1+R_2R_3(5R_0R_4+R_1R_4V_0)}{R_AR_B-R_0R_1R_2R_3} &= V_3 \qquad (2)
\end{aligned}
$$

Sanity checks: Equations 1 and 2 are in units of *Volts* (which is what we want). Also, if $R_4 = 0$, then equation 1 reduces to $V_2 = V_3$ (which is what we expect from the circuit diagram). Likewise, if $R_3 = 0$, then $V_3 = 0$. If $R_2 = 0$, then $V_3 = V_1$. If $R_0 = 0$, then $V_2 = V_0$. If $R_1 = 0$, then $V_2 = 5$.

3. (10pts) Assume $R_0 = 100\Omega$, $R_1 = 200\Omega$, $R_2 = 300\Omega$, $R_3 = 400\Omega$, $R_4 = 100\Omega$ and $V_1 = +8V$. Show a plot of I_4 as V_0 is varied from $-10V$ to $+10V$. You are encouraged to use a tool such as Matlab to produce this plot.

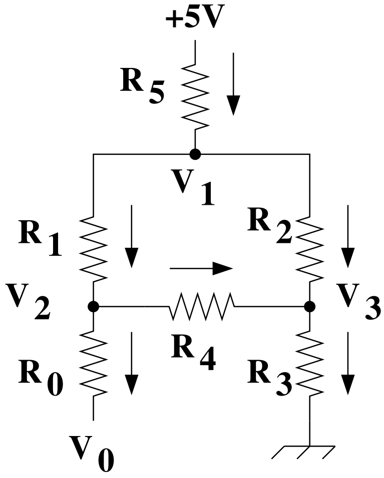


4. (10pts) Given the same assumptions as above, show a plot of V_2 and V_3 as V_0 is varied from $-10V$ to $+10V$ (place both of these curves on the same plot). Describe any interesting relationships between this plot and the previous one.



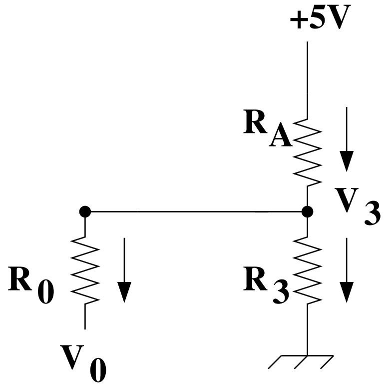
Question 3

Consider the following circuit:



Assume that V_0 and R_i are given.

1. (20pts) Assuming that $R_4 = 0\Omega$, show the simplest equivalent circuit. You may use your knowledge of the series/parallel resistor rules.



Where $R_A = R_5 + \frac{R_1 R_2}{R_1+R_2}$.

2. (10pts) List the fundamental equations that are derived directly from the simplified circuit (note that you should have the same number of equations as unknowns).

   These are always true:

$$\begin{aligned}
5 - V_3 &= I_A R_A \\
V_3 - 0 &= I_3 R_3 \\
V_3 - V_0 &= I_0 R_0 \\
I_A &= I_0 + I_3
\end{aligned}$$

3. (20pts) Derive equations for V_1, V_2 and V_3 for the simplified circuit. One should be a function of only known variables; the others may also assume knowledge of those that have already been derived.

$$V_2 = V_3$$

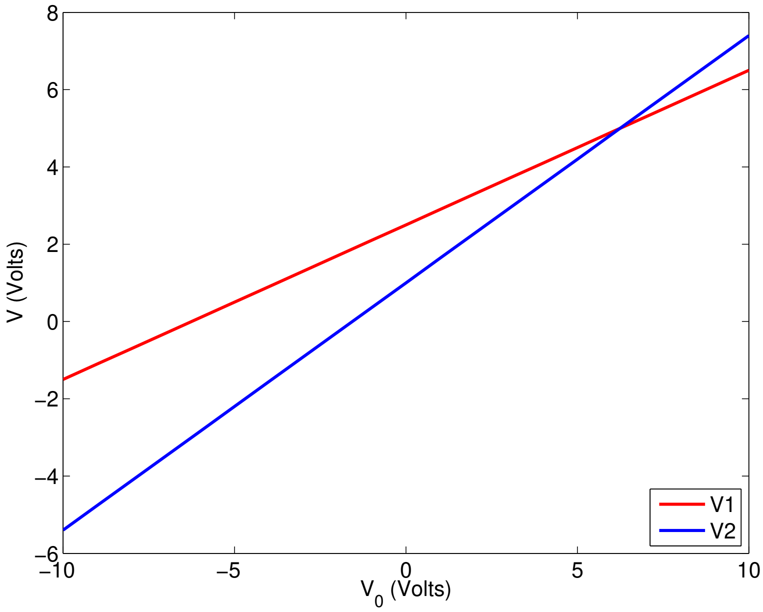
$$\begin{aligned}
\frac{5 - V_3}{R_A} &= \frac{V_3 - V_0}{R_0} + \frac{V_3}{R_3} \\
5R_0R_3 - V_3R_0R_3 &= V_3R_AR_3 - V_0R_AR_3 + V_3R_AR_0 \\
5R_0R_3 + V_0R_AR_3 &= V_3(R_AR_3 + R_0R_3 + R_AR_0) \\
\frac{5R_0R_3 + V_0R_AR_3}{R_B} &= V_3
\end{aligned}$$

$$I_A = \frac{5 - V_3}{R_A}$$

$$I_5 = I_A$$

$$\begin{aligned}
5 - V_1 &= I_5 R_5 \\
V_1 &= 5 - I_5 R_5 \\
&= 5 - (5 - V_3)\frac{R_5}{R_A}
\end{aligned}$$

4. (10pts) Assume $R_0 = 100\Omega$, $R_1 = 200\Omega$, $R_2 = 300\Omega$, $R_3 = 400\Omega$ and $R_5 = 200\Omega$ (again, for the simplified circuit). Show a plot of V_1 and V_2 as V_0 is varied from $-10V$ to $+10V$.



Graduate Only

5. (10pts) Given the original circuit, list the fundamental equations that are derived directly from the circuit (note that you should have the same number of equations as unknowns).

   These are always true:

$$\begin{aligned}
I_5 &= I_1 + I_2 \\
I_1 &= I_0 + I + 4 \\
I_4 + I_2 &= I_3 \\
I_1 &= \frac{V_1 - V_2}{R_1} \\
I_2 &= \frac{V_1 - V_3}{R_2} \\
I_3 &= \frac{V_3}{R_3} \\
I_4 &= \frac{V_2 - V_3}{R_4} \\
I_5 &= \frac{5 - V_1}{R_5}
\end{aligned}$$

6. (20pts) Derive equations for V_1, V_2 and V_3. One should be a function of only known variables; the others may also assume knowledge of those that have already been derived.

   The first of Kirchoff's equations:

$$\begin{aligned}
\frac{5 - V_1}{R_5} &= \frac{V_1 - V_2}{R_1} + \frac{V_1 - V_3}{R_2} \\
5R_1R_2 - R_1R_2V_1 &= R_2R_5V_1 - R_2R_5V_2 + R_1R_5V_1 - R_1R_5V_3 \\
R_A &\equiv R_2R_5 + R_1R_2 + R_1R_5 \\
5R_1R_2 &= R_AV_1 - R_2R_5V_2 - R_1R_5V_3 \qquad (3)
\end{aligned}$$

   Note that R_A has units of Ω^2.

The second of Kirchoff's equations:

$$\begin{aligned}
\frac{V_1 - V_2}{R_1} &= \frac{V_2 - V_0}{R_0} + \frac{V_2 - V_3}{R_4} \\
V_1 R_0 R_4 - V_2 R_0 R_4 &= V_2 R_1 R_4 + V_0 R_1 R_4 + V_2 R_0 R_1 - V_3 R_0 R_1 \\
V_1 R_0 R_4 + V_0 R_1 R_4 + V_3 R_0 R_1 &= V_2 (R_0 R_4 + R_1 R_4 + R_0 R_1) \\
R_B &\equiv R_0 R_4 + R_1 R_4 + R_0 R_1 \\
V_1 R_0 R_4 + V_0 R_1 R_4 + V_3 R_0 R_1 &= V_2 R_B \qquad (4)
\end{aligned}$$

Note that R_B also has units of Ω^2.
The third of Kirchoff's equations:

$$\begin{aligned}
\frac{V_2 - V_3}{R_4} &= \frac{V_1 - V_3}{R_2} + \frac{V_3}{R_3} \\
V_2 R_2 R_3 - V_3 R_2 R_3 + V_1 R_3 R_4 - V_3 R_3 R_4 &= V_3 R_2 R_4 \\
V_2 R_2 R_3 + V_1 R_3 R_4 &= V_3 (R_2 R_3 + R_3 R_4 + R_2 R_4) \\
R_C &\equiv R_2 R_3 + R_3 R_4 + R_2 R_4 \\
V_2 R_2 R_3 + V_1 R_3 R_4 &= V_3 R_C \qquad (5)
\end{aligned}$$

Note that R_C also has units of Ω^2.
Now substitute equation 5 into 3:

$$\begin{aligned}
5 R_1 R_2 R_C &= V_1 R_A R_C - R_2 R_5 R_C V_2 - R_1 R_2 R_3 R_5 V_2 - R_1 R_3 R_4 R_5 V_1 \\
5 R_1 R_2 R_C &= V_1 (R_A R_C - R_1 R_3 R_4 R_5) - V_2 (R_2 R_5 R_c + R_1 R_2 R_3 R_5) \\
R_E &\equiv R_2 R_5 R_c + R_1 R_2 R_3 R_5 \\
5 R_1 R_2 R_C &= V_1 (R_A R_C - R_1 R_3 R_4 R_5) - V_2 R_E \qquad (6)
\end{aligned}$$

Note that R_E has units of Ω^4.
Now substitute equation 5 into 4:

$$\begin{aligned}
V_1 R_0 R_4 R_C + V_0 R_1 R_4 R_C + (V_2 R_2 R_3 + V_1 R_3 R_4) R_0 R_1 &= V_2 R_B R_C \\
(R_0 R_4 R_C + R_0 R_1 R_3 R_4) V_1 + V_0 R_1 R_4 R_C &= V_2 (R_B R_C - R_0 R_1 R_2 R_3) \\
R_D &\equiv R_B R_C - R_0 R_1 R_2 R_3 \\
(R_0 R_4 R_C + R_0 R_1 R_3 R_4) V_1 + V_0 R_1 R_4 R_C &= V_2 R_D \qquad (7)
\end{aligned}$$

R_D has units of Ω^4.

Now substitute equation 6 into 7:

$$\begin{aligned}
5R_1R_2R_CR_D &= (R_AR_CR_D - R_1R_3R_4R_5R_D)V_1 \\
&\quad -(R_2R_5R_C + R_1R_2R_3R_5) \\
&\quad \times [V_1(R_0R_4R_C + R_0R_1R_3R_4) + V_0R_1R_4R_C] \\
R_E &\equiv R_2R_5R_C + R_1R_2R_3R_5 \\
5R_1R_2R_CR_D + V_0R_1R_4R_CR_E &= V_1(R_AR_CR_D - R_1R_3R_4R_5R_D \\
&\quad -R_0R_4R_CR_E + R_0R_1R_3R_4R_E) \\
R_F &\equiv R_AR_CR_D - R_1R_3R_4R_5R_D - R_0R_4R_CR_E \\
&\quad +R_0R_1R_3R_4R_E \\
V_1 &= \frac{5R_1R_2R_CR_D + V_0R_1R_4R_CR_E}{R_F} \qquad (8)
\end{aligned}$$

R_E has units of Ω^4 and R_F has units of Ω^8.

Sanity checks:

- $R_0 = 0 \Rightarrow V_2 = V_0$
- $R_1 \to 0 \Rightarrow V_1 \to V_2$ (setting the resistance to zero yields undefined values for the V's in our general solution)
- $R_2 \to 0 \Rightarrow V_1 \to V_3$
- $R_3 = 0 \Rightarrow V_3 = 0$
- $R_4 \to 0 \Rightarrow V_2 \to V_3$
- $R_5 = 0 \Rightarrow V_1 = 5$

7. (10pts) Assume $R_0 = 100\Omega$, $R_1 = 200\Omega$, $R_2 = 300\Omega$, $R_3 = 400\Omega$, $R_4 = 100\Omega$ and $R_5 = 200\Omega$. Show a plot of V_1, V_2 and V_3 as V_0 is varied from $-10V$ to $+10V$.