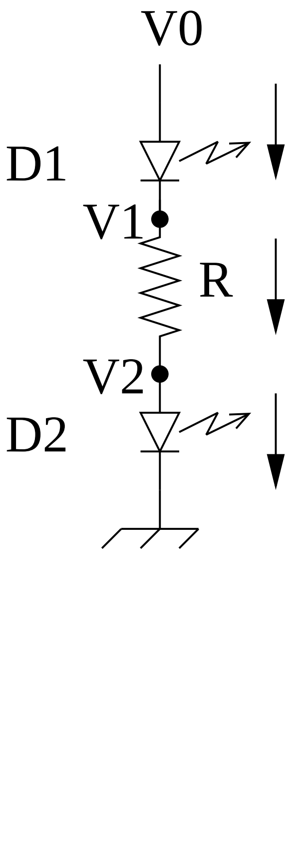
Embedded Systems (CS [45]163)
Homework 2 Solutions

March 10, 2010

Question 1

Consider the following circuit:



1. (10pts) What properties are **always** true for this circuit? You may assume that the following are known: $V0$, R, V_{f1} and V_{f2}.

$$
\begin{aligned}
I_1 &= I_R \\
I_R &= I_2 \\
V_1 - V_2 &= I_R R
\end{aligned}
$$

2. (20pts) Derive equations for the currents and intermediate voltages as a function of the known variables. Make sure to address all cases and determine the ranges of $V0$ for which they apply.

   Hint: the state of a resistor that is not connected to a circuit is technically undetermined. We can assume that current through the resistor is zero in this case, but the voltages are not constrained (but note that in the Q1 circuit, the voltages are constrained by the diodes, even when current is not flowing).

   There are four cases.

   Case 1: diode 1 is turned on and diode 2 is turned off.

   This implies that:

$$
\begin{aligned}
V_0 - V_1 &= V_{f1} \\
I_1 &> 0 \\
I_2 &= 0
\end{aligned}
$$

   Which results in: $0 = I_2 = I_1 > 0$. This can never happen. Therefore, this case will never happen.

   Case 2: diode 1 is turned off and diode 2 is turned on.

   The logic (and conclusion) is the same as in case 1.

   Case 3: both diodes are turned on.

$$
V_0 - V_1 = V_{f1}
$$

$$
\begin{aligned}
I_1 &> 0 \\
V_2 - 0 &= V_{f2} \\
I_2 &> 0
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
V_1 &= V_0 - V_{f1} \\
V_2 &= V_{f2} \\
I_R &= \frac{V_0 - V_{f1} - V_{f2}}{R}
\end{aligned}
$$

Note that I_R must be greater than zero. This implies that:

$$
V_0 > V_{f1} + V_{f2}
$$

Case 4: both diodes are turned off.

This implies that:

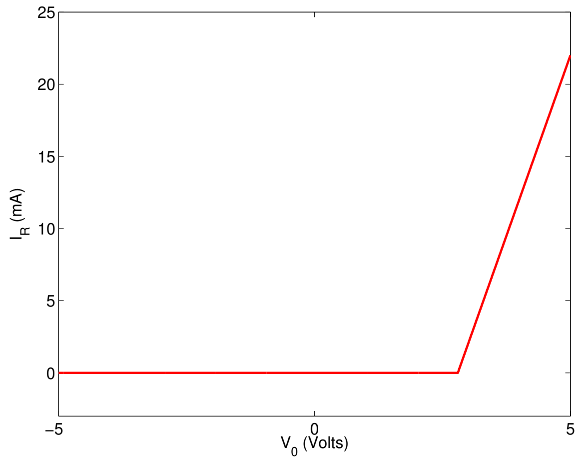
$$
\begin{aligned}
I_1 &= 0 \\
V_0 - V_1 &< V_{f1} \\
I_2 &= 0 \\
V_2 - 0 &< V_{f2}
\end{aligned}
$$

Because $I_R = 0$, $V_1 = V_2$. However, V_1 is undetermined. All we know is that:

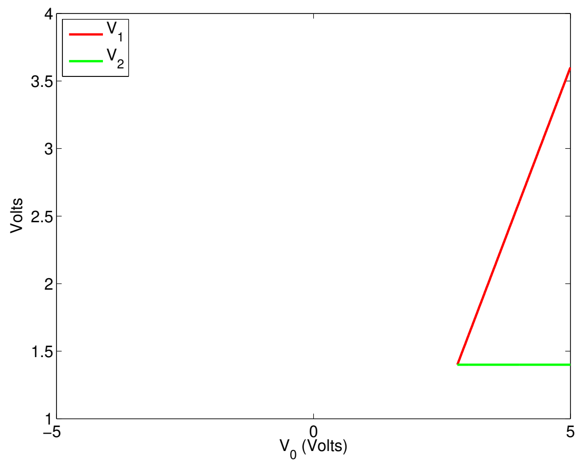
$$
\begin{aligned}
V_0 - V_1 + V_2 &< V_{f1} + V_{f2} \\
V_0 &< V_{f1} + V_{f2}
\end{aligned}
$$

3. (10pts) Assume that $V_{f1} = V_{f2} = 1.4V$. Show I_R as a function of $V0$ (which varies from $-5V$ to $5V$).

I assume that $R = 100\Omega$.

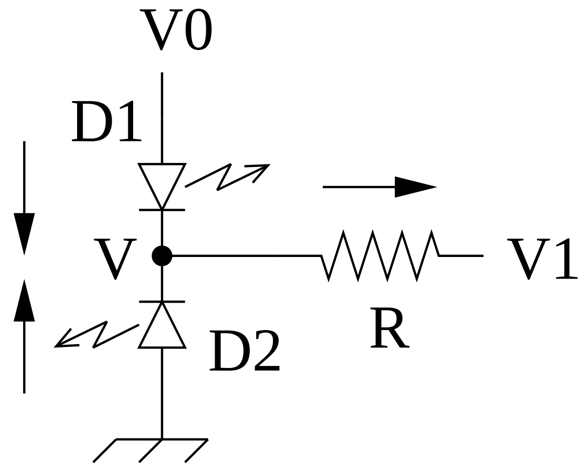


4. (10pts) Show $V1$ and $V2$ as a function of $V0$ (which varies from $-5V$ to $5V$).



Question 2

Consider the following circuit:



1. (10pts) What properties are **always** true for this circuit? You may assume that the following are known: $V0$, $V1$, R, V_{f1} and V_{f2}.

$$\begin{aligned} I_1 + I2 &= I_R \\ V - V_1 &= I_R R \end{aligned}$$

2. (20pts) Derive equations for the currents and V. Make sure to address all cases and determine the ranges of $V0$ and $V1$ for which they apply.

   Case 1: both diodes are turned off.

   This implies that:

$$\begin{aligned} I_1 &= 0 \\ V_0 - V &< V_{f1} \\ I_2 &= 0 \\ 0 - V &< V_{f2} \end{aligned}$$

Because $I_R = 0$, we know that $V = V_1$.

Therefore, we know that:

$$\begin{aligned} V_1 &> -V_{f2} \\ V_0 - V_1 &< V_{f1} \end{aligned}$$

Case 2: D2 is on, D1 is off.

This implies that:

$$\begin{aligned} I_1 &= 0 \\ V_0 - V &< V_{f1} \\ I_2 &> 0 \\ 0 - V &= V_{f2} \end{aligned}$$

Therefore:

$$\begin{aligned} I_R &= \frac{-V_{f2} - V_1}{R} \\ V_1 &< -V_{f2} \\ V_0 &< V_{f1} - V_{f2} \end{aligned}$$

Case 3: D1 is on, D2 is off.

This implies that:

$$\begin{aligned} I_1 &> 0 \\ V_0 - V &= V_{f1} \\ I_2 &= 0 \\ 0 - V &< V_{f2} \end{aligned}$$

Therefore:

$$\begin{aligned} V &= V_0 - V_{f1} \\ -V_0 + V_{f1} &< V_{f2} \\ V_0 &> V_{f1} - V_{f2} \\ I_R &= \frac{V_0 - V_{f1} - V_1}{R} \\ V_1 &< V_0 - V_{f1} \\ V_0 - V_1 &> V_{f1} \end{aligned}$$

Case 4: Both are on. This implies that:

$$\begin{aligned} I_1 &> 0 \\ V_0 - V &= V_{f1} \\ I_2 &> 0 \\ 0 - V &= V_{f2} \end{aligned}$$

Therefore:

$$\begin{aligned} V_0 &= V_{f1} - V_{f2} \qquad (1) \\ V &= -V_{f2} \\ I_R &= \frac{-V_{f2} - V_1}{R} \\ V_1 &< -V_{f2} \end{aligned}$$

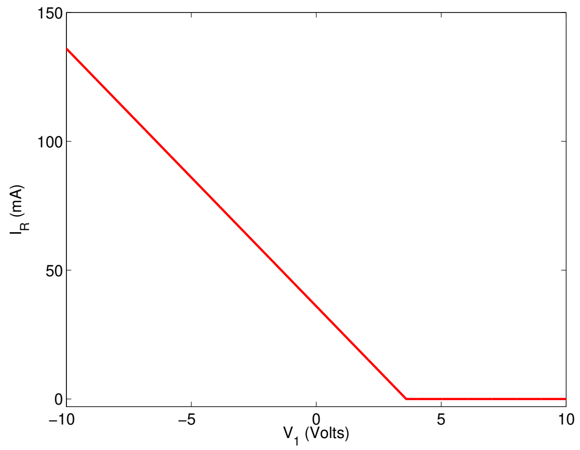
Note that equation 1 implies that there is a constraint in this case on the possible value of V_0 (we can't choose it arbitrarily). While this is true in our ideal analysis, the reality is that some small amount of resistance in the diodes and the wires will allow V_0 in this case.

(in retrospect, I should have inserted the extra resistor between V_0 and $D1$, but this would have made the analysis more complex)

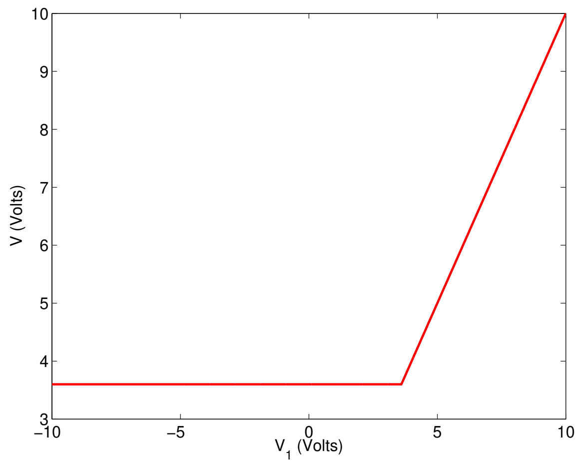
3. (10pts) Assume that $V_{f1} = V_{f2} = 1.4V$ and $V0 = 5V$. Show I_R as a function of $V1$ (which varies from $-10V$ to $10V$).

   Note 1: I assumed $R = 100\Omega$.

   Note 2: For $V_0 = 5V$, we can only have to cases: both off and diode 1 on only.

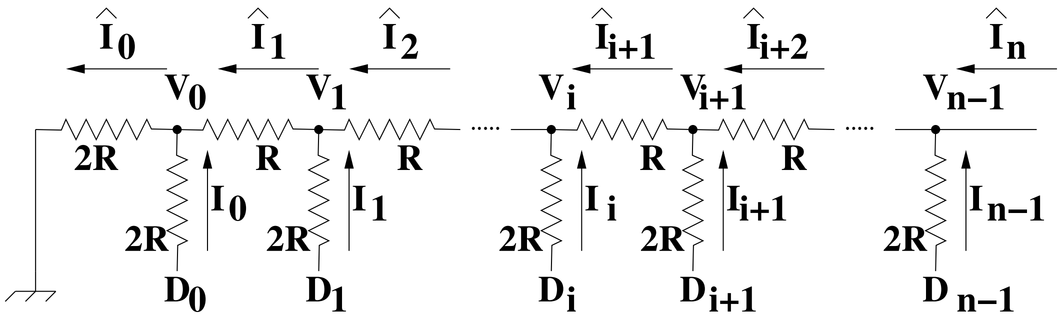


4. (10pts) Show V as a function of $V1$ (which varies from $-10V$ to $10V$).



Question 3

Consider the following digital-to-analog conversion circuit. Note that there are only two types of resistors in this network. The output is V_{n-1}. Our task is to solve for the output given the binary digital inputs.



Note that $\hat{I}_n$ is given for mathematical convenience, but you should assume that it is equal to zero. $D_i \in \{0, 1\}$, so you can assume that the voltage at the pin is $5D_i$.

1. (10pts) List the equations that are given to us by Ohm's law. Be as generic as possible.

$$\begin{aligned} 5D_i - V_i &= 2RI_i \\ V_0 - 0 &= 2R\hat{I}_0 \\ V_i - V_{i-1} &= R\hat{I}_i \end{aligned}$$

2. (10pts) List the equations that are given to us by Kirchhoff's current law.

$$\hat{I}_i = I_i + \hat{I}_{i+1}$$

3. (30pts) Solve for V_{n-1} as a function of $D_0 ... D_{n-1}$ (there will be no other parameters or variables). Undergraduates: you may assume that $n = 3$; Graduates: solve for a generic $n > 2$.

   Hint: Start by solving for V_0 as a function of D_0 and $\hat{I}_1$ (and there will be an R term as well). Then solve for V_1 as a function of D_0, D_1, and $\hat{I}_2$. Follow the recursion from there.

   Taking $i = 0$ and substituting the first two Ohm's law equations into Kirchhoff's equation:

$$\begin{aligned} \frac{V_0}{2R} &= \frac{5D_0 - V_0}{2R} + \hat{I}_1 \\ V_0 &= \frac{5D_0}{2} + R\hat{I}_1 \end{aligned}$$

   Note: if we assume that $\hat{I}_1 = 0$ (i.e., that the rest of the circuit is removed), then we are left with a voltage divider circuit (with which our solution is consistent).

   Before we take the next step, let's rearrange one of the Ohm's law equations:

$$V_{i-1} = V_i - R\hat{I}_i$$

We can now use this equation to advance from V_i to V_{i+1}. In this case:

$$\begin{aligned} V_1 - R\hat{I}_1 &= \frac{5D_0}{2} + R\hat{I}_1 \\ V_1 &= \frac{5D_0}{2} + 2R\hat{I}_1 \end{aligned}$$

Our Kirchhoff's equation allows us to advance from $\hat{I}_i$ to $\hat{I}_{i+1}$. In this case:

$$V_1 = \frac{5D_0}{2} + 2R\left(I_1 + \hat{I}_2\right)$$

Finally, bringing in our other Ohm's law equation:

$$\begin{aligned} V_1 &= \frac{5D_0}{2} + 2R\left(\frac{5D_1 - V_1}{2R} + \hat{I}_2\right) \\ 2V_1 &= \frac{5D_0}{2} + 5D_1 + 2R\hat{I}_2 \\ V_1 &= \frac{5D_0}{4} + \frac{5D_1}{2} + R\hat{I}_2 \end{aligned}$$

And taking the next step:

$$V_2 = \frac{5D_0}{8} + \frac{5D_1}{4} + \frac{5D_2}{2} + R\hat{I}_3$$

For $n = 3$, $\hat{I}_3 = 0$. Therefore:

$$V_2 = \frac{5D_0}{8} + \frac{5D_1}{4} + \frac{5D_2}{2}$$

(and the undergrads are done).

For arbitrary n, we can keep working our way down the V's:

$$2^n V_{n-1} = 5 \sum_{i=0}^{n-1} 2^i D_i + 2^n R \hat{I}_n$$

But: $\hat{I}_n = 0$. Therefore:

$$\begin{aligned} 2^n V_{n-1} &= 5 \sum_{i=0}^{n-1} 2^i D_i \\ V_{n-1} &= 5 \sum_{i=0}^{n-1} 2^{i-n} D_i \end{aligned}$$